

ON J_* -NOETHERIAN RINGS

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ABSTRACT. In this paper, we introduce and study the class of J^* -Noetherian rings. A commutative ring R is defined to be J^* -Noetherian if its Jacobson radical $J(R)$ is a Noetherian R -module, meaning every ideal in $J(R)$ is finitely generated. This generalizes the concept of Noetherian rings, and we explore several properties and characterizations of J^* -Noetherian rings. We present examples to demonstrate the distinction between Noetherian rings and J^* -Noetherian rings, proving that not all J^* -Noetherian rings are Noetherian. Additionally, we extend classical results such as the Eakin-Nagata-Formanek theorem to the context of J^* -Noetherian rings. We also investigate the stability of the J^* -Noetherian property under various ring extensions, such as polynomial and power series extensions, as well as trivial ring extensions. Furthermore, we provide homological characterizations of J^* -Noetherian rings, including a Cartan-Eilenberg-Bass type theorem, which states that a ring is J^* -Noetherian if and only if direct sums and direct limits of J^* -injective modules remain J^* -injective. Our results provide new insights into the structure of rings via the lens of their Jacobson radical.

1. INTRODUCTION

Throughout this paper, we focus solely on commutative rings with identity. R will always denote such a ring. The concept of Noetherian rings is one of the most important topics and is widely used in many areas, including commutative algebra and algebraic geometry. The Noetherian property was first introduced by the mathematician Emmy Noether, who established a connection between the ascending chain condition on ideals and the property of being finitely generated. More precisely, she showed that for a ring R , the ascending chain condition on ideals holds if and only if every ideal of R is finitely generated. This equivalence plays a significant role in simplifying the structure of ideals in a ring.

The importance of the Noetherian property was first demonstrated in Hilbert's Basis Theorem, which states that a ring R is Noetherian if and only if the polynomial ring $R[X]$ (and, correspondingly, the formal power series ring $R[[X]]$) is also Noetherian. Noetherian rings also have many module-theoretic characterizations, such as the well-known Cartan-Eilenberg-Bass

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Theorem, which states that a ring R is Noetherian if and only if every direct sum of injective R -modules is injective, and equivalently, if every direct limit of injective R -modules over a directed set is injective (see [16, Theorem 4.3.4]).

Due to the significance of Noetherian rings, many mathematicians have explored Noetherian properties in various classes of rings and have sought to generalize the notion of Noetherian rings. One famous generalization is the concept of coherent rings, i.e., rings in which every finitely generated ideal is finitely presented.

For a further generalization, Ding et al. [9] introduced the notion of J -coherent rings in terms of the Jacobson radical ideal in 2009. A ring R is said to be J -coherent provided that $J(R)$ is a coherent R -module in the sense of [6], i.e., any finitely generated ideal in $J(R)$ is finitely presented (note that the definition of a coherent module is restricted to finitely generated modules in [15, Definition 4.51]). On the other hand, in [5], Dabbabi and Benhissi introduced the notion of non- J -Noetherian rings, also in terms of the Jacobson radical ideal. A ring R is called non- J -Noetherian if each non- J -ideal (i.e., an ideal not contained in the Jacobson radical of R) is finitely generated.

The main motivation of this paper is to introduce and study the J_* -Noetherian property of rings. Compared with the concepts of Noetherian rings and J -coherent rings, we define a ring R to be J_* -Noetherian if $J(R)$ is a Noetherian R -module, or equivalently, if every ideal in $J(R)$ is finitely generated. Trivially, Noetherian rings are J_* -Noetherian. Several examples and counterexamples are provided in Example 2.2. We also establish the Eakin-Nagata-Formanek Theorem for J_* -Noetherian rings (see Theorem 2.3).

In Theorem 2.9, we prove that if R is a J_* -Noetherian ring, then $R[X]$ is also a J_* -Noetherian ring. However, Example 2.10 demonstrates that the converse is not true in general. Subsequently, we explore the stability of this concept in power series extensions with a finite number of indeterminates. In Corollary 2.16, we show that the power series ring $R[[X_1, \dots, X_n]]$ is J_* -Noetherian if and only if R is a Noetherian ring.

It is natural to inquire about the relationship between J -coherent rings and J_* -Noetherian rings. To address this, we study the transfer of the J_* -Noetherian property via trivial ring extensions. We show that the trivial ring extension $R \ltimes M$ is J_* -Noetherian if and only if R is J_* -Noetherian and M is a Noetherian R -module (see Theorem 2.18).

Using these results, we provide three counterexamples. First, we find a coherent (and thus J -coherent) ring that is not J_* -Noetherian (see Example 2.22). Second, in Example 2.23, we construct a J -coherent ring that is neither coherent nor J_* -Noetherian. Finally, surprisingly, J_* -Noetherian rings can also fail to be J -coherent (see Example 2.24).

The final section is dedicated to providing some homological characterizations of J_* -Noetherian and J -coherent rings in terms of J_* -injective, J -injective, and J -flat modules. Notably, we show that direct limits of injective modules need not be J -injective for J_* -Noetherian rings (see Remark 3.4).

Let $J(R)$, $\text{Nil}(R)$, and $Z(R)$ respectively denote the Jacobson radical of R , the nilradical of R , and the set of all zero-divisors of R . For any terms and notations not defined in this paper, readers are referred to [16] for further clarification.

2. ON J_* -NOETHERIAN RINGS

We start with the following definition.

Definition 2.1. A ring R is said to be a J_* -Noetherian ring provided that any ideal in $J(R)$ is finitely generated.

Recall from [17] that a ring R is said to be a $\text{Nil}(R)_*$ -Noetherian ring if every nil ideal (i.e., ideal in $\text{Nil}(R)$) is finitely generated.

Example 2.2. (1) Every Noetherian ring is J_* -Noetherian.

(2) Obviously, if $J(R) = (0)$, then R is J_* -Noetherian, since the only ideal in $J(R)$ is 0, which is finitely generated.

(3) A local ring (R, M) is J_* -Noetherian if and only if it is Noetherian. In particular, if R is a chained ring.

(4) The converse of (1) fails, i.e., there exist J_* -Noetherian rings that are not Noetherian. In fact, any non-Noetherian zero-dimensional reduced ring (such as a von Neumann regular ring) is J_* -Noetherian.

(5) A ring R is Noetherian if and only if R is J_* -Noetherian and non- J -Noetherian [5, Proposition 1].

(6) Each J_* -Noetherian ring is Nil_* -Noetherian.

(7) The converse of (6) fails, i.e., there exist J_* -Noetherian rings that are not Noetherian. In fact, any non-Noetherian local domain is a Nil_* -Noetherian ring that is not J_* -Noetherian.

(8) If $J(R) = \text{Nil}(R)$, then R is J_* -Noetherian if and only if it is Nil_* -Noetherian. In particular, if R is a zero-dimensional ring, then R is J_* -Noetherian if and only if it is Nil_* -Noetherian.

The first result of this paper is to provide the Eakin-Nagata-Formanek Theorem for J_* -Noetherian rings.

Theorem 2.3. *Let R be a ring. Then the following statements are equivalent:*

- (1) *Every nonempty family of ideals in $J(R)$ has a maximal element;*
- (2) *R is J_* -Noetherian;*
- (3) *Every ascending chain of ideals in $J(R)$ is stationary.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal in $J(R)$. Set Ω to be the set of all finitely generated ideals that are included in I . For every $a \in I$, the fact that $aR \subseteq I \subseteq J(R)$ ensures that $aR \in \Omega$, and hence Ω is nonempty. By

assumption, Ω has a maximal element L . On the other hand, L is finitely generated. Write $L = x_1R + \cdots + x_nR$. Now, our aim is to prove that $I = L$. Let $\alpha \in I$ and set $Q = L + \alpha R$. Therefore, $Q \subseteq I$ and Q is a finitely generated ideal of R , so $Q \in \Omega$. Since $L \subseteq Q$, by the maximality of L , we have $Q \subseteq L$. Therefore, $\alpha \in L$, and thus $I = L$ is finitely generated.

(2) $\Rightarrow$ (3) Let $(I_n)_{n \in \mathbb{N}}$ be an ascending chain of ideals in $J(R)$. If $I = \bigcup_{n \in \mathbb{N}} I_n$, then I is an ideal of R in $J(R)$, and by hypothesis, I is finitely generated. This implies that $I = Ra_1 + \cdots + Ra_p$ for some $a_1, \dots, a_p \in I$. Hence, there exists $k \in \mathbb{N}$ such that $I \subseteq I_k$. Therefore, $I_n = I_k$ for any $n \geq k$, and thus $(I_n)_{n \in \mathbb{N}}$ is stationary.

(3) $\Rightarrow$ (1) Let Γ be a nonempty set of ideals in $J(R)$. Suppose that Γ has no maximal elements. If we take any $I_1 \in \Gamma$, then I_1 is not a maximal element. Thus, there exists $I_2 \in \Gamma$ such that $I_1 \subset I_2$. Since I_2 is not a maximal element, we can find $I_3 \in \Gamma$ such that $I_2 \subset I_3$. Hence, we obtain an ascending chain $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ of ideals in $J(R)$. This chain is not stationary. $\square$

It is known in [13, Problem 1, page 52] that if a ring R satisfies the ascending chain condition on finitely generated ideals, then R is Noetherian. We have the following similar result for J_* -Noetherian rings.

Proposition 2.4. *Let R be a ring. If R satisfies the ascending chain condition on finitely generated ideals in $J(R)$, then R is a J_* -Noetherian ring.*

Proof. Assume that there is an ideal I in $J(R)$ that is not finitely generated. Let $a_1 \in I$. Then $(a_1) \subset I$. Let $a_2 \in I \setminus (a_1)$. Then $(a_1) \subset (a_1, a_2) \subset I, \dots$ forms a strictly ascending chain of finitely generated ideals in $J(R)$, contradicting the ascending chain condition. $\square$

Proposition 2.5. *Let R be a J_* -Noetherian ring. If I is an ideal in $J(R)$, then R/I is J_* -Noetherian.*

Proof. Let K be an ideal in $J(R/I)$. Then $K = J/I$ for some R -ideal J containing I . Let $\mathfrak{m}$ be a maximal ideal of R . Since $I \subseteq J(R) \subseteq \mathfrak{m}$, we conclude that $\mathfrak{m}/I$ is a maximal ideal of R/I . Hence, $K = J/I \subseteq \mathfrak{m}/I$, and thus $J \subseteq \mathfrak{m}$ for every $\mathfrak{m} \in \text{Max}(R)$. Therefore, $J \subseteq J(R)$, meaning J is finitely generated. Hence, K is a finitely generated R/I -ideal. $\square$

Note that the condition “ I is in $J(R)$ ” in Proposition 2.5 cannot be removed.

Example 2.6. Let $A = D[X_1, X_2, \dots]$ be the polynomial ring over a local domain $(D, \mathfrak{m})$ with countably infinite variables. Then A is J_* -Noetherian since $J(A) = 0$. Now, consider the quotient ring $R = A / \langle X_i^2 \mid i \geq 1 \rangle$. It is easy to see that $J(R) = \mathfrak{m} + \langle \overline{X_1}, \overline{X_2}, \dots \rangle$ is infinitely generated, where $\overline{X_i}$ denotes the representative of X_i in R for each i . Hence, R is not J_* -Noetherian.

Proposition 2.7. *A finite direct product of rings $R = R_1 \times \cdots \times R_n$ is J_* -Noetherian if and only if each R_i is J_* -Noetherian ($i = 1, \dots, n$).*

Proof. This follows from the fact that $J(R) = J(R_1) \times \cdots \times J(R_n)$, and $J(R)$ is a Noetherian R -module if and only if each $J(R_i)$ is a Noetherian R_i -module ($i = 1, \dots, n$). $\square$

The well-known Hilbert Basis Theorem states that a ring R is Noetherian if and only if $R[X]$ is Noetherian (see [16, Theorem 4.3.15]).

Lemma 2.8. ([16, Exercise 1.47]) *Let R be a ring. Then $J(R[X]) = \text{Nil}(R[X]) = \text{Nil}(R)[X]$.*

Theorem 2.9. *Let R be a ring. If R is a J_* -Noetherian ring, then $R[X]$ is a J_* -Noetherian ring.*

Proof. Assume that R is a J_* -Noetherian ring. Then R is also Nil_* -Noetherian, and hence $R[X]$ is a J_* -Noetherian ring by combining Lemma 2.8 with [17, Theorem 1.9]. $\square$

The following example shows that the converse of Theorem 2.9 does not hold.

Example 2.10. Let D be a local domain that is not Noetherian. Then $J(R[X]) = 0$ according to Lemma 2.8, and thus $R[X]$ is a J_* -Noetherian ring. However, R itself is not J_* -Noetherian.

Remark 2.11. Let $\phi : R \rightarrow S$ be a ring homomorphism making R a module retract of S . If S is J_* -Noetherian, then R is not necessarily J_* -Noetherian, as noted in the above example. This shows that the well-known result “every retract of a Noetherian ring is Noetherian” does not generalize to J_* -Noetherian rings.

Surprisingly, unlike in the classical case, the localization of a J_* -Noetherian ring can be non- J_* -Noetherian.

Example 2.12. Take a field k and the ring $R = k[X_1, \dots, X_n, \dots]$. Since $J(R) = 0$, R is a J_* -Noetherian ring. However, the localization of R at the maximal ideal $\mathfrak{m} = (X_1, \dots, X_n, \dots)$ is not (J_-) Noetherian, as the chain $(X_1) \subset (X_1, X_2) \subset \cdots \subset (X_1, \dots, X_n) \subset \cdots$ forms a strictly ascending chain of ideals in $R_{\mathfrak{m}}$.

Proposition 2.13. *Let R be Nil_* -Noetherian and S a multiplicative subset of R . Then R_S is also Nil_* -Noetherian.*

Proof. By replacing the descending chain with an ascending chain, the proof is similar to that of [18, Proposition 2.5], thus we omit it. $\square$

It is also well-known that a ring R is Noetherian if and only if $R[[X]]$ is Noetherian (see [16, Theorem 4.3.15]).

Lemma 2.14. (1) *If I is an ideal of R , then I is a finitely generated ideal of R if and only if $I[[X]]$ is a finitely generated ideal of $R[[X]]$.*

- (2) If J is an ideal of $R[[X]]$, then J is finitely generated if and only if XJ is finitely generated.

Proof. (1) Apply [14, Lemma 2] in the specific case $S = \{1\}$.

(2) This follows from the fact that $XJ \cong J$ (since X is a regular element of $R[[X]]$). $\square$

Theorem 2.15. *Let R be a ring. Then $R[[X]]$ is a J_* -Noetherian ring if and only if R is a J_* -Noetherian ring.*

Proof. For necessity, assume that $R[[X]]$ is a J_* -Noetherian ring. Let I be an ideal of R . Since $XI[[X]] \subseteq J(R) + XR[[X]] = J(R[[X]])$, we conclude that $XI[[X]]$ is a finitely generated ideal of $R[[X]]$. Hence, $I[[X]]$ is a finitely generated ideal of $R[[X]]$ according to Lemma 2.14 (2), and therefore, I is a finitely generated ideal of R by Lemma 2.14 (1). For sufficiency, see [16, Theorem 4.3.15]. $\square$

By induction, we have the following result.

Corollary 2.16. *Let R be a ring. Then $R[[X_1, \dots, X_n]]$ is a J_* -Noetherian ring if and only if R is a Noetherian ring.*

Example 2.17. Let R be a non-Noetherian reduced ring (i.e., $\text{Nil}(R) = 0$). Then $R[[X]]$ is a Nil_* -Noetherian ring, but it is not J_* -Noetherian.

Let A be a ring and let M be an A -module. Then, the set $A \ltimes M$, consisting of all pairs $(r, m) \in A \times M$ with componentwise addition and multiplication defined by $(r, m)(b, f) = (rb, rf + bm)$, is a unitary commutative ring, called the trivial extension (or idealization) of A by M . Recall from [3, Theorem 4.8] that the trivial ring extension $R \ltimes M$ is Noetherian if and only if R is Noetherian and M is finitely generated. In particular, trivial ring extensions have been useful in solving many open problems and conjectures in both commutative and non-commutative ring theory. The basic properties of the trivial ring extension are summarized in [3, 12].

Lemma 2.18. ([3, Theorem 3.2]) *Let R be a commutative ring and M an R -module. Then the Jacobson radical of the trivial ring extension $A \ltimes M$ is*

$$J(A \ltimes M) = J(A) \ltimes M.$$

Now we determine when the trivial ring extension $R \ltimes M$ is J_* -Noetherian.

Theorem 2.19. *Let R be a ring and M an R -module. Then the following assertions are equivalent:*

- (1) $R \ltimes M$ is a J_* -Noetherian ring.
- (2) R is a J_* -Noetherian ring and M is a Noetherian R -module.

Proof. (1) $\Rightarrow$ (2) Since $R \cong R \ltimes M / 0 \ltimes M$ according to [3, Theorem 3.1], and $0 \ltimes M \subseteq J(R) \ltimes M = J(R \ltimes M)$ by Lemma 2.18, R is J_* -Noetherian by Proposition 2.5. Since $0 \ltimes N$ is an ideal in $J(R \ltimes M)$ for every submodule N of M , it follows that $0 \ltimes N$ is a finitely generated $R \ltimes M$ -module. Hence,

it is easy to check that N is also a finitely generated R -module. Thus, M is a Noetherian R -module.

(2) $\Rightarrow$ (1) Suppose R is a J_* -Noetherian ring and M is a Noetherian R -module. Note that we have the following exact sequence of $(R \ltimes M)$ -modules:

$$0 \rightarrow 0 \ltimes M \xrightarrow{i} R \ltimes M \xrightarrow{\pi} R \rightarrow 0.$$

Let $J^\bullet : J_1 \subseteq J_2 \subseteq \cdots$ be an ascending chain of ideals in $J(R \ltimes M)$. Then there is an ascending chain of ideals in $J(R)$:

$$\pi(J^\bullet) : \quad \pi(J_1) \subseteq \pi(J_2) \subseteq \cdots.$$

Thus, there exists $k \in \mathbb{Z}^+$ such that $\pi(J_k) = \pi(J_n)$ for any $n \geq k$. Similarly,

$$J^\bullet \cap 0 \ltimes M : \quad J_1 \cap (0 \ltimes M) \subseteq J_2 \cap (0 \ltimes M) \subseteq \cdots$$

is an ascending chain of sub-ideals of $0 \ltimes M$. For every $i \geq 1$, there exists a submodule N_i of M such that $J_i \cap (0 \ltimes M) = 0 \ltimes N_i$. This implies that $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of M . Thus, there exists $k' \in \mathbb{Z}^+$ such that $N_k = N_n$ for any $n \geq k'$, and therefore $J_n \cap 0 \ltimes M = J_k \cap 0 \ltimes M$. Let $l = \max\{k, k'\}$ and $n \geq l$. Consider the following natural commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_l \cap (0 \ltimes M) & \longrightarrow & J_l & \longrightarrow & \pi(J_l) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & J_n \cap (0 \ltimes M) & \longrightarrow & J_n & \longrightarrow & \pi(J_n) \longrightarrow 0 \end{array}$$

Thus, we have $J_n = J_l$ for any $n \geq l$. Therefore, $R \ltimes M$ is a J_* -Noetherian ring by Theorem 2.3. $\square$

The following example illustrates two points: firstly, that the condition “ M is a Noetherian R -module” in Theorem 2.19 cannot be weakened to “ M is a finitely generated R -module,” and secondly, that [17, Theorem 2.1] is not generally true.

Example 2.20. Let R be a von Neumann regular ring which is not Noetherian. Then R is a J_* -Noetherian ring (since $J(R) = \text{Nil}(R) = 0$), and $M := R$ is a finitely generated R -module. However, $R \ltimes M$ is not a J_* -Noetherian ring, according to Theorem 2.19.

Now, we determine when the trivial ring extension $R \ltimes M$ is Nil_* -Noetherian.

Theorem 2.21. *Let R be a ring and M an R -module. Then the following assertions are equivalent:*

- (1) $R \ltimes M$ is a Nil_* -Noetherian ring;
- (2) R is a Nil_* -Noetherian ring and M is a Noetherian R -module.

Proof. The proof is similar to that of Theorem 2.19, and thus we omit it. $\square$

An R -module M is called divisible if $aM = M$ for every non-zero-divisor $a \in R$ (see [16, Definition 1.6.10(2)]). It is easy to see that every quotient module of a divisible module is divisible; every direct sum and every direct product of divisible modules is also divisible. Additionally, the quotient field K of a domain R is a divisible R -module.

Let I be an ideal of R , and let J be a nonempty subset of R . The residual of I by J is defined as $(I : J) = \{x \in R \mid xJ \subseteq I\}$. Furthermore, for $J = (a) \subseteq R$, we prefer the notation $(I : a)$ instead of $(I : (a))$.

Recall from [9] that a ring R is called J -coherent provided that any finitely generated ideal in $J(R)$ is finitely presented. The next example proves that J -coherent rings are not J_* -Noetherian in general.

Example 2.22. The trivial ring extension $R = \mathbb{Z} \ltimes \mathbb{Q}/\mathbb{Z}$ is a coherent ring that is not J_* -Noetherian.

Proof. According to Theorem 2.19, R is not J_* -Noetherian since $\mathbb{Q}/\mathbb{Z}$ is not a Noetherian $\mathbb{Z}$ -module (in fact, it is not finitely generated). However, we will show that R is coherent. Indeed, let $0 \neq x \in R$. Then two cases are possible:

Case 1: If $x \in J(R)$, then there exist nonzero $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ such that $x = (0, \frac{a}{b} + \mathbb{Z})$ by Lemma 2.18. Hence, it is easy to check that $(0 : x) = b\mathbb{Z} \ltimes \mathbb{Q}/\mathbb{Z}$ is a principal ideal of R according to [4, Lemma 2.2].

Case 2: If $x \notin J(R)$, then $x = (d, e)$ for some $0 \neq d \in \mathbb{Z}$ and $e \in \mathbb{Q}/\mathbb{Z}$. Thus,

$$(0 :_R x) = 0 \ltimes \left(\frac{1}{d} + \mathbb{Z} \right) = (0, \frac{1}{d})R$$

is a principal ideal of R . Thus, in both cases, we conclude that $(0 : x)$ is a principal ideal of R (and therefore finitely generated).

On the other hand, let J_1 and J_2 be two finitely generated ideals of R . We claim that $J_1 \cap J_2$ is also finitely generated. Three cases are possible:

Case 1: If both ideals J_1 and J_2 are not in $J(R)$, since $J(R)$ is a divided ideal of R , we conclude that there exist two ideals $I_1 = n_1\mathbb{Z}$ and $I_2 = n_2\mathbb{Z}$ such that $J_i = I_i \ltimes \mathbb{Q}/\mathbb{Z}$ for $i = 1, 2$ (see [3, Corollary 3.4] and recall that $\mathbb{Q}/\mathbb{Z}$ is a divisible $\mathbb{Z}$ -module). Hence, $J_1 \cap J_2 = (I_1 \cap I_2) \ltimes \mathbb{Q}/\mathbb{Z}$ is a principal ideal of R by [4, Lemma 2.2].

Case 2: If one ideal is in $J(R)$ but the other is not, without loss of generality, assume that $J_1 \subseteq J(R)$ and J_2 is not in $J(R)$. Since $\mathbb{Q}/\mathbb{Z}$ is a divisible $\mathbb{Z}$ -module, we conclude that every ideal of R is comparable with $J(R)$ according to [3, Corollary 3.4]. Thus, $J(R) \subseteq J_2$, and therefore $J_1 \cap J_2 = J_1$ is finitely generated.

Case 3: If both ideals are in $J(R)$, then there exist two submodules F_1 and F_2 of $\mathbb{Q}/\mathbb{Z}$ such that $J_1 = 0 \ltimes F_1$ and $J_2 = 0 \ltimes F_2$. Since J_1 and J_2 are finitely generated ideals of R , we can easily see that F_1 and F_2 are finitely generated submodules of $\mathbb{Q}/\mathbb{Z}$. Thus, $F_1 + F_2$ is finitely generated, and therefore $F_1 + F_2$ is a finitely presented $\mathbb{Z}$ -module (as a finitely generated

module over a Noetherian ring). Hence, $F_1 \cap F_2$ is a finitely generated $\mathbb{Z}$ -module according to [10, Corollary 2.1.3]. Thus, $(0 \ltimes F_1) \cap (0 \ltimes F_2) = 0 \ltimes (F_1 \cap F_2)$ is a finitely generated ideal of R . Hence, our claim is true, and thus R is coherent according to [10, Theorem 2.13]. $\square$

By making a slight alteration to the previous example, we can construct a J -coherent ring that is neither coherent nor J_* -Noetherian.

Example 2.23. Let $E := \bigoplus_{n \geq 0} E_n$ with $E_n := \mathbb{Q}/\mathbb{Z}$, and set $R = \mathbb{Z} \ltimes E$. Then, by Theorem 2.19, R is not J_* -Noetherian since E is not a Noetherian $\mathbb{Z}$ -module. On the other hand, R is not coherent according to [1, Example 2.6]. Now we claim that R is J -coherent. Let $x \in J(R)$. Then $x = (0, e)$ for some $e \in E$. As for every $e \in E$, there exists $0 \neq d \in \mathbb{Z}$ such that $de = 0$, we have that $(0 :_R x) = \text{Ann}(e) \ltimes E$ is a principal ideal of R (and thus finitely generated) by [4, Lemma 2.2]. On the other hand, using the same approach as in the third case of the previous example, we find that the intersection of any two finitely generated ideals in $J(R)$ is finitely generated. Thus, R is J -coherent according to [9, Theorem 2.13].

Remarkably, in contrast to the classical case, J_* -Noetherian rings can also be non- J -coherent.

Example 2.24. Set $A = \mathbb{Z}[X_1, X_2, \dots]$, the polynomial ring over $\mathbb{Z}$ with countably infinite variables. Then the natural projection $\pi : A \rightarrow \mathbb{Z}$ defines an A -module structure on $\mathbb{Z}$. The trivial ring extension $R := A \ltimes \mathbb{Z}$ is J_* -Noetherian but not J -coherent.

Proof. Since $J(R) = 0 \ltimes \mathbb{Z}$, every ideal of R in $J(R)$ has the form $0 \ltimes F$, where F is an A -submodule of $\mathbb{Z}$. As the restriction of π on $\mathbb{Z}$ is the identity, we conclude that the submodules of $\mathbb{Z}$ as an A -module are exactly the submodules of $\mathbb{Z}$ as a $\mathbb{Z}$ -module (i.e., ideals of $\mathbb{Z}$). Hence, $0 \ltimes F = 0 \ltimes n\mathbb{Z} = (0, n)R$ is a principal ideal of R . Thus, R is a J_* -Noetherian ring. However, the fact that $r = (0, 1) \in J(R)$ and $(0 : r) = \langle X_1, X_2, \dots \rangle \ltimes \mathbb{Z}$ is infinitely generated implies that R is not J -coherent according to [9, Theorem 2.13]. $\square$

Theorem 2.25. *Let R be a ring such that $J(R)$ is a regular ideal. Then:*

- (1) *R is a J_* -Noetherian ring if and only if R is a Noetherian ring.*
- (2) *R is a J -coherent ring if and only if R is a coherent ring.*

Proof. (1) If R is a Noetherian ring, then it is naturally J_* -Noetherian. Conversely, assume that R is a J_* -Noetherian ring and let I be a proper ideal of R . We claim that I is finitely generated. Indeed, let a be a regular element in $J(R)$. Then $aI \subseteq aR \subseteq J(R)$, and so aI is a finitely generated ideal of R (since $aI \subseteq J(R)$ and R is a J_* -Noetherian ring). It follows that I is a finitely generated ideal of R since $aI \cong I$ (as a is a regular element of R). Therefore, R is a Noetherian ring.

- (2) The proof is similar to (1), and so we omit it. $\square$

Remark 2.26. Let R be a J_* -Noetherian ring. Then R is a J -coherent ring if $J(R)$ is a regular ideal according to Theorem 2.25. Hence, if R is a J_* -Noetherian ring that is not J -coherent, then $J(R) \subseteq Z(R)$. Furthermore, if R is J_* -Noetherian with $Z(R) = J(R)$, then we can easily see that R is J -coherent. In summary, a J_* -Noetherian ring R is not J -coherent if and only if there exists $x \in J(R) \subsetneq Z(R)$ such that $(0 : x)$ is an infinitely generated non- J -ideal.

3. MODULE-THEORETIC CHARACTERIZATIONS OF J_* -NOETHERIAN AND J -COHERENT RINGS

Let R be a ring. Recall from [9] that an R -module M is called J -injective if $\text{Ext}_R^1(R/I, M) = 0$ for every finitely generated ideal I in $J(R)$, or equivalently, if for any finitely generated ideal I in $J(R)$, every homomorphism $\alpha : I \rightarrow M$ extends to a homomorphism $\beta : R \rightarrow M$. An R -module N is said to be J -flat if $\text{Tor}_1(N, R/I) = 0$ for every finitely generated ideal I in $J(R)$.

In what follows, $\mathcal{JI}$ (resp. $\mathcal{JF}$) stands for the class of all J -injective R -modules (resp. J -flat R -modules). We can easily observe that $\mathcal{JI}$ (resp. $\mathcal{JF}$) is closed under extensions, direct products (resp. direct limits), direct sums, and direct summands. Moreover, the classes $\mathcal{JI}$ and $\mathcal{JF}$ are closed under pure submodules (see [9, Lemma 2.4]).

We begin this section with the following concepts.

Definition 3.1. An R -module M is said to be J_* -injective provided that $\text{Ext}_R^1(R/I, M) = 0$ for any ideal I in $J(R)$, or equivalently, if for any ideal I in $J(R)$, every homomorphism $\alpha : I \rightarrow M$ extends to a homomorphism $\beta : R \rightarrow M$.

Trivially, injective modules are J_* -injective, and the class of J_* -injective modules is closed under direct products.

Theorem 3.2. (*Cartan-Eilenberg-Bass Theorem for J_* -Noetherian rings*)
Let R be a ring. Then the following assertions are equivalent:

- (1) R is J_* -Noetherian;
- (2) Each direct union of J_* -injective (resp., injective) modules is J_* -injective;
- (3) Each direct sum of J_* -injective (resp., injective) modules is J_* -injective;
- (4) For every J_* -injective (resp., injective) R -module E , $\bigoplus E$ is J_* -injective;

Proof. (1) $\Rightarrow$ (2) Let $\{E_i, f_{i,j}\}_{i < j, i, j \in A}$ be a direct system of J_* -injective modules, where each $f_{i,j}$ is the embedding map. Let $\varinjlim E_i$ be its direct limit. Let I be an ideal in $J(R)$. Then it is finitely generated. Since R/I is finitely presented, we have

$$\text{Ext}_R^1(R/I, \varinjlim E_i) \cong \varinjlim \text{Ext}_R^1(R/I, E_i) = 0$$

by Lenzing's theorem. Thus, $\varinjlim E_i$ is J_* -injective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) These implications are straightforward.

(3) $\Rightarrow$ (1) Suppose, for the sake of contradiction, that R is not J_* -Noetherian. Hence, there exists a non-finitely generated ideal I in $J(R)$. So there is an element $a_0 \in I$ such that $a_0R \neq I$. Take $0 \neq a_1 \in I \setminus a_0R$, then the ideal $a_1R + a_0R \neq I$. Take $a_2 \in I \setminus (a_1R + a_0R)$, then the ideal $a_1R + a_2R + a_0R \neq I$. Repeating these steps, we get a strictly increasing chain of ideals in $J(R)$:

$$a_0R + a_1R \subsetneq a_0R + a_1R + a_2R \subsetneq \cdots \subsetneq a_0R + a_1R + \cdots + a_nR \subsetneq \cdots$$

Set $A_i = \sum_{j=0}^i a_jR$. Using [2, Corollary 10.5], we conclude that for any A_i , there exists a maximal sub-ideal C_i of A_i satisfying

$$C_1 \subsetneq A_1 \subseteq C_2 \subsetneq A_2 \subseteq \cdots \subseteq C_n \subsetneq A_n \subseteq \cdots$$

Set $\Omega = \{A_i/C_i\}$. Denote by $E(F)$ the injective envelope of F for each $F \in \Omega$. Then $\bigoplus_{F \in \Omega} E(F)$ is J_* -injective by (2). Set $A = \bigcup_{i=1}^{\infty} A_i$. Then A is an ideal in $J(R)$. Set $\pi_i : A_i \rightarrow A_i/C_i$ as the natural epimorphism and $\rho_i : A_i/C_i \hookrightarrow E(A_i/C_i)$ as the inclusion map. Then there are homomorphisms $f_i : A \rightarrow E(A_i/C_i)$ which are extensions of $\rho_i \circ \pi_i$. Let $a \in A$, and let $f : A \rightarrow \bigoplus_{F \in \Omega} E(F)$ satisfy $f(a) = (f_i(a))_{i=1}^{\infty}$. Since $\bigoplus_{F \in \Omega} E(F)$ is J_* -injective, f can be extended to a homomorphism $g : R \rightarrow \bigoplus_{F \in \Omega} E(F)$ with $g(r) = g(1)r$. Set $g(1) = (c_1, c_2, \dots, c_n, 0, \dots)$. Take $a \in A$ such that $a \in A_{n+1} \setminus C_{n+1}$. Then $\rho_{n+1}\pi_{n+1}(a) \neq 0$, and so $f_{n+1}(a) \neq 0$. However,

$$f(a) = g(a) = g(1)a = (c_1a, c_2a, \dots, c_na, 0, \dots)$$

which is a contradiction. Thus, R is J_* -Noetherian.

(4) $\Rightarrow$ (3) Let $\{E_i\}_{i \in \Omega}$ be a family of J_* -injective modules. Then $\prod_{i \in \Omega} E_i$ is J_* -injective. Hence, $\bigoplus (\prod_{i \in \Omega} E_i)$ is J_* -injective by assumption. Since $\bigoplus_{i \in \Omega} E_i$ is a direct summand of $\bigoplus (\prod_{i \in \Omega} E_i)$, we conclude that $\bigoplus_{i \in \Omega} E_i$ is J_* -injective. $\square$

Corollary 3.3. *Let R be a ring. Then R is J_* -Noetherian if and only if any J -injective module is J_* -injective.*

Proof. Assume that R is a J_* -Noetherian ring. Let I be an ideal in $J(R)$ and M a J -injective module. Then I is finitely generated, and thus R/I is finitely presented. It follows that $\text{Ext}_R^1(R/I, M) = 0$. Consequently, M is J_* -injective.

Conversely, assume that any J -injective module is J_* -injective. Since the class of J -injective modules is closed under direct sums, R is a J_* -Noetherian ring by Theorem 3.2. $\square$

Remark 3.4. We must remark that a ring R is J_* -Noetherian if and only if any direct union of (J_-) -injective R -modules is J_* -injective, as stated in Theorem 3.2. Thus, if any direct limit of injective R -modules is J_* -injective, then R is a J_* -Noetherian ring. However, the converse does not hold in general. Indeed, let R be a J_* -Noetherian ring that is not J -coherent (see Example 2.24). Then the class of J_* -injective R -modules is equal to that of

J -injective modules. However, by [9, Theorem 2.13], there exists a direct system of (J_*) -injective R -modules whose direct limit is not J_* -injective.

In 1993, Chen and Ding in [8] showed that a ring R is coherent if and only if $\text{Hom}_R(M, E)$ is flat for any absolutely pure R -module M and injective R -module E , and if and only if $\text{Hom}_R(M, E)$ is flat for any injective R -modules M and E . We generalize this result to J -coherent rings.

Theorem 3.5. *The following assertions are equivalent for a ring R .*

- (1) R is a J -coherent ring;
- (2) $\text{Hom}_R(M, E)$ is J -flat for every J -injective module M and every injective module E ;
- (3) $\text{Hom}_R(\text{Hom}_R(M, E_1), E_2)$ is J -flat for every J -flat module M and every injective module E_1, E_2 ;
- (4) If E_1 and E_2 are injective cogenerators, then $\text{Hom}_R(\text{Hom}_R(M, E_1), E_2)$ is J -flat for every J -flat module M .

We need the following lemma of independent interest before proving Theorem 3.5.

Lemma 3.6. *Let R be a ring and M be an R -module. Then the following assertions are equivalent:*

- (1) M is J -flat;
- (2) $\text{Hom}_R(M, E)$ is J -injective for any injective R -module E ;
- (3) $\text{Hom}_R(M, E)$ is a J -injective module for any injective cogenerator R -module E .

Proof. (1) $\Rightarrow$ (2) Let I be a finitely generated ideal in $J(R)$ and E an injective R -module. Since M is J -flat, it follows from [16, Theorem 3.4.11] that $\text{Ext}_R^1(R/I, \text{Hom}_R(M, E)) \cong \text{Hom}_R(\text{Tor}_1^R(R/I, M), E) = 0$. Thus, $\text{Hom}_R(M, E)$ is J -injective.

(2) $\Rightarrow$ (3) This is straightforward.

(3) $\Rightarrow$ (1) Let I be a finitely generated ideal in $J(R)$ and E an injective cogenerator R -module. Since $\text{Hom}_R(M, E)$ is J -injective, it follows that $\text{Hom}_R(\text{Tor}_1^R(R/I, M), E) \cong \text{Ext}_R^1(R/I, \text{Hom}_R(M, E)) = 0$. Since E is an injective cogenerator R -module, we can deduce that $\text{Tor}_1^R(R/I, M) = 0$. Therefore, M is J -flat. $\square$

Proof of Theorem 3.5. (1) $\Rightarrow$ (2) Let I be a finitely generated ideal in $J(R)$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Tor}_1^R(\text{Hom}_R(M, E), R/I) & \longrightarrow & \text{Hom}_R(M, E) \otimes I & \longrightarrow & \text{Hom}_R(M, E) \\
 & & \downarrow \psi_{R/I} & & \downarrow \psi_I & & \downarrow \psi_R \\
 0 & \longrightarrow & \text{Hom}(\text{Ext}_R^1(R/I, M), E) & \longrightarrow & \text{Hom}(\text{Hom}(I, M), E) & \longrightarrow & \text{Hom}(\text{Hom}(R, M), E)
 \end{array}$$

Since I and R are finitely presented and E is injective, ψ_I and ψ_R are isomorphisms by [16, Theorem 2.6.13(2)]. Thus, $\psi_{R/I}$ is an isomorphism by the

Five Lemma. On the other hand, since M is J -injective, $\text{Ext}_R^1(R/I, M) = 0$, hence $\text{Tor}_1^R(\text{Hom}_R(M, E), R/I) = 0$. Therefore, $\text{Hom}_R(M, E)$ is J -flat.

(2) $\Rightarrow$ (3) This follows from Lemma 3.6.

(3) $\Rightarrow$ (4) This is straightforward.

(4) $\Rightarrow$ (1) Let $\{M_i\}_{i \in \Gamma}$ be a family of J -flat modules and E_1 and E_2 be injective cogenerators. Then $\bigoplus_{i \in \Gamma} M_i$ is J -flat. Thus,

$$\text{Hom}_R \left(\text{Hom}_R \left(\bigoplus_{i \in \Gamma} M_i, E_1 \right), E_2 \right) \cong \text{Hom}_R \left(\prod_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2 \right)$$

is J -flat by hypothesis. Since $\bigoplus_{i \in \Gamma} \text{Hom}_R(M_i, E_1)$ is a pure submodule of $\prod_{i \in \Gamma} \text{Hom}_R(M_i, E_1)$ by [7, Lemma 1(1)], the natural epimorphism:

$$\text{Hom}_R \left(\prod_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2 \right) \rightarrow \text{Hom}_R \left(\bigoplus_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2 \right)$$

splits by [11, Lemma 2.19]. Hence,

$$\prod_{i \in \Gamma} \text{Hom}_R(\text{Hom}_R(M_i, E_1), E_2) \cong \text{Hom}_R \left(\bigoplus_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2 \right)$$

is J -flat. Note from [11, Corollary 2.21] that $\prod_{i \in \Gamma} M_i$ is a pure submodule of $\prod_{i \in \Gamma} \text{Hom}_R(\text{Hom}_R(M_i, E_1), E_2)$. Thus, $\prod_{i \in \Gamma} M_i$ is J -flat by [9, Lemma 2.4]. Hence, R is J -coherent by [9, Theorem 2.12]. $\square$

By combining [19, Lemma 3.4], [9, Theorem 2.13], and [9, Lemma 2.4], we can deduce the following results:

Proposition 3.7. *Let R be a ring. The following statements are equivalent:*

- (1) R is J -coherent;
- (2) The class of J -injective R -modules is closed under pure quotients;
- (3) The class of J -injective R -modules is closed under direct limits;
- (4) The class of J -injective R -modules is precovering;
- (5) The class of J -injective R -modules is covering.

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